# Arf closure versus strict closure

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based on the works jointly with

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# 1. Introduction

Let

- $S/R$ an extension of commutative rings
- $\overline{R}$ the integral closure of $R$ in $\mathrm{Q}(R)$.

We define

$$R \subseteq R^* = \{x \in S \mid x \otimes 1 = 1 \otimes x \text{ in } S \otimes_R S\} \subseteq S$$

and we say that

- $R$ is *strictly closed in $S$*, if $R = R^*$ holds in $S$.
- $R$ is *strictly closed*, if $R = R^*$ holds in $\overline{R}$.

Notice that

- $(R^*)^* = R^*$ in $S$
- $R^* \subseteq T^*$ in $S$ for all $R \subseteq T \subseteq S$.

## Example 1.1

Let $S = k[X, Y]$ be the polynomial ring over a field $k$.

(1) Let $n \geq 3$ and set

$$R = k[X^{n-i}Y^i \mid 0 \leq i \leq n, \ i \neq 1].$$

Then $R$ is a strictly closed ring with $\dim R = 2$.

(2) Let $R = k[X^4, XY^3, Y^4]$. Then $R^* = k[X^4, XY^3, X^7Y^5, Y^4]$ in $\overline{R}$.

## Example 1.2

Let $(R, \mathfrak{m})$ be a RLR with $\dim R = 2$. Let $\mathfrak{m} = (x, y)$, $I = (x^3, xy^4, y^5)$. Then the Rees algebra

$$\mathcal{R}(I) = R[It]$$

is strictly closed, where $t$ is an indeterminate.

**Example 1.3**

Let $S = k[[t]]$ be the formal power series ring over a field $k$. Consider

$$R = k[[t^3, t^8, t^{13}]] \subseteq T = k[[t^3, t^5]] \subseteq S.$$

Then $R$ is NOT strictly closed in $S = \overline{R}$, but it is strictly closed in $T$.

- In 1949, Cahit Arf explored the multiplicity sequences of curve singularities.
- In 1971, J. Lipman defined "Arf rings" for one-dimensional CM semi-local rings.

**Definition 1.4 (Lipman, 1971)**

Let $R$ be a CM semi-local ring with $\dim R = 1$. Then $R$ is called *an Arf ring*, if the following hold:

(1) Every integrally closed *open* ideal $I$ has a principal reduction.

(2) If $x, y, z \in R$ s.t.

$$x \text{ is a NZD on } R \text{ and } \frac{y}{x}, \ \frac{z}{x} \in \overline{R},$$

then $yz/x \in R$.

Notice that

(1) $I^{n+1} = aI^n$ for $\exists\ n \geq 0$ and $\exists\ a \in I$.

(2) Stability of $I$ (if reduction exists).

Hence

**Theorem 1.5 (Lipman, 1971)**

*Let $R$ be a CM semi-local ring with* $\dim R = 1$. *Then*

$$R \text{ is Arf} \iff \text{Every integrally closed open ideal is stable.}$$

When $R$ is a CM local ring with $\dim R = 1$,

if $R$ is an Arf ring, then $R$ has minimal multiplicity.

We assume

- $(R, \mathfrak{m})$ is a Noetherian complete local domain with $\dim R = 1$
- $R/\mathfrak{m}$ is an algebraically closed field of characteristic 0

Lipman proved:

$R$ is *saturated* $\implies$ $R$ has minimal multiplicity.

Moreover, among all Arf rings between $R$ and $\overline{R}$,

$\exists$ the smallest one $\mathrm{Arf}(R)$, called Arf closure.

Lipman extends the results of C. Arf about multiplicity sequences.

**Proposition–Definition 1.6**

*Let $R$ be a CM semi-local ring with $\dim R = 1$. Suppose $\overline{R}$ is a finitely generated $R$-module. Then, among all Arf rings between $R$ and $\overline{R}$, there is the smallest Arf ring $\mathrm{Arf}(R)$, called the Arf closure of $R$.*

**Conjecture 1.7 (Zariski, 1971)**

Let $R$ be a CM semi-local ring with $\dim R = 1$. Suppose $\overline{R}$ is a finitely generated $R$-module. Then the equality

$$\mathrm{Arf}(R) = R^*$$

holds in $\overline{R}$.

- Zariski's conjecture holds if $R$ contains a field (Lipman).

**Theorem 1.8 (Main result)**

*Zariski's conjecture holds.*

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### Theorem 1.8 (Main result)

*Zariski's conjecture holds.*

# 2. Proof of Zariski's conjecture

### Theorem 2.1

*Let $R$ be a CM semi-local ring with $\dim R = 1$. Then TFAE.*

(1) *$R$ is a strictly closed ring.*

(2) *$R$ is an Arf ring.*

### known results

Let $R$ be a CM semi-local ring with $\dim R = 1$. Then

- $R$ is strictly closed $\Longrightarrow$ $R$ is Arf. (Zariski)
- The converse holds if $R$ contains a field. (Lipman)

# Proof of $(2) \Rightarrow (1)$

There is a filtration:

$$R \subseteq J : J \subseteq J^2 : J^2 \subseteq \cdots \subseteq J^m : J^m \subseteq \cdots \subseteq \overline{R}$$

where $J$ denotes the Jacobson radical of $R$. Define

$$R \ \subseteq \ R^J = \bigcup_{m \geq 0} [J^m : J^m] \ \subseteq \ \overline{R}.$$

For $n \geq 0$, we set

$$R_n = \begin{cases} R & \text{if } n = 0 \\ R_{n-1}^{J(R_{n-1})} & \text{if } n \geq 1 \end{cases}$$

where $J(R_{n-1})$ stands for the Jacobson radical of $R_{n-1}$.

Hence

$$R \subseteq R_1 \subseteq \cdots \subseteq R_n \subseteq \cdots \subseteq \overline{R}.$$

**Step 1**

The equality $\overline{R} = \bigcup_{n \geq 0} R_n \ (= \varinjlim R_n)$ holds.

**Step 2**

The equality $R = R^*$ holds in $R_n$ for $\forall n \geq 0$.

**Lemma 2.2 (Key lemma)**

*Let $(R, \mathfrak{m})$ be a CM local ring with $\dim R = 1$. Suppose that $\mathfrak{m}^2 = z\mathfrak{m}$ for some $z \in \mathfrak{m}$. Let $R_1 \subseteq C \subseteq \overline{R}$ be an intermediate ring s.t. C is a finitely generated R-module and let*

$$\alpha : C \otimes_R C \to C \otimes_{R_1} C$$

*be an R-algebra map s.t. $\alpha(x \otimes y) = x \otimes y$ for $\forall x, y \in C$. Then*

$$\operatorname{Ker} \alpha = (0) :_{C \otimes_R C} z$$

*holds.*

Let $x \in R^*$ in $\overline{R}$ and choose $n \geq 0$ such that $x \in R_n$. Since $\overline{R} = \varinjlim R_m$, we get

$$\begin{array}{rcl} \overline{R} \otimes_R R_n & \to & \overline{R} \otimes_R \overline{R} = \varinjlim (\overline{R} \otimes_R R_m) \\ x \otimes 1 - 1 \otimes x & \mapsto & 0. \end{array}$$

There exists $\ell \geq n$ such that

$$\overline{R} \otimes_R R_n \to \overline{R} \otimes_R R_\ell, \quad x \otimes 1 - 1 \otimes x \mapsto 0.$$

Since

$$\begin{array}{rcl} R_n \otimes_R R_\ell & \to & \overline{R} \otimes_R R_\ell = \varinjlim (R_m \otimes_R R_\ell) \\ x \otimes 1 - 1 \otimes x & \mapsto & 0, \end{array}$$

there exists $p \geq n$ such that

$$R_n \otimes_R R_\ell \to R_p \otimes_R R_\ell, \quad x \otimes 1 - 1 \otimes x \mapsto 0.$$

For $q \in \mathbb{Z}$ such that $q \geq p$ and $q \geq \ell$, we obtain

$$\begin{array}{ccccc} R_p \otimes_R R_\ell & \to & R_p \otimes_R R_\ell & \to & R_q \otimes_R R_q \\ x \otimes 1 - 1 \otimes x & \mapsto & 0 & \mapsto & 0 \end{array}$$

Therefore

$$x \in R_n \subseteq R_q \quad \text{and} \quad x \otimes 1 = 1 \otimes x \quad \text{in } R_q \otimes_R R_q$$

so that $x \in R^*$ in $R_q$. Thus $x \in R$. Hence $R = R^*$ in $\overline{R}$. □

**Theorem 2.3**

*Let $R$ be a CM semi-local ring with $\dim R = 1$. Then*

$$R \text{ is strictly closed} \iff R \text{ is Arf.}$$

*Hence, $\mathrm{Arf}(R) = R^*$ holds, provided $\overline{R}$ is a finitely generated $R$-module.*

### Theorem 2.4

*Let $R$ be a CM semi-local ring with $\dim R = 1$. Then*

$$R \text{ is Arf} \implies R^G \text{ is Arf}$$

*for every finite subgroup $G$ of $\operatorname{Aut} R$ s.t. the order of $G$ is invertible.*

# 3. Strictly closed rings

**Question 3.1**

What kind of rings are strictly closed?

## Theorem 3.2

*Let $R$ be a commutative ring and $T$ an $R$-subalgebra of $\mathrm{Q}(R)$. Let $V$ be a non-empty subset of $T$ s.t. $T = R[V]$. If $fg \in R$ for all $f, g \in V$, then $R$ is strictly closed in $T$.*

## Corollary 3.3

*Let $R$ be a commutative ring and $J = (a_1, a_2, \ldots, a_n)$ $(n \geq 3)$ an ideal of $R$ s.t. $a_1^2 = a_2 a_3$. Set $I = (a_2, a_3, \ldots, a_n)$ and consider*

$$\mathcal{R} = \mathcal{R}(I) \subseteq \mathcal{T} = \mathcal{R}(J)$$

*Then $\mathcal{R}$ is strictly closed in $\mathcal{T}$, provided $I$ contains a NZD on $R$.*

## Theorem 3.4

*The Stanley-Reisner ring $R = k[\Delta]$ of $\Delta$ is strictly closed.*

**Thank you for your attention.**